1. $\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\sin\theta} r \, dr \, d\theta$

$= \int_{\theta=0}^{\theta=\pi} \left[\frac{1}{2} r^2\right]_{r=0}^{r=\sin\theta} d\theta$

$= \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \left((\sin\theta)^2 - 0^2\right) d\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \sin^2\theta \, d\theta$

$= \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \frac{1-\cos 2\theta}{2} \, d\theta$

$= \frac{1}{2} \cdot \frac{1}{2} \int_{\theta=0}^{\theta=\pi} (1-\cos 2\theta) \, d\theta$

$= \frac{1}{4} \left[\int_{\theta=0}^{\theta=\pi} 1 \, d\theta - \int_{\theta=0}^{\theta=\pi} \cos 2\theta \, d\theta \right]$

$u = 2\theta \quad du = 2 \, d\theta$

$= \frac{1}{4} \left[[\theta]_0^{\pi} - \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \cos 2\theta \, 2 \, d\theta \right]$

$= \frac{1}{4} \left[\pi - 0 - \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \cos u \, du \right]$

$= \frac{1}{4} \left[\pi - \frac{1}{2} \left[\sin u \right]_{\theta=0}^{\theta=\pi} \right]$

$= \frac{1}{4} \left[\pi - \frac{1}{2} \left[\sin 2\theta \right]_{\theta=0}^{\theta=\pi} \right]$

$= \frac{1}{4} \left[\pi - \frac{1}{2} \left[\sin 2\pi - \sin 2(0) \right] \right]$

$= \frac{1}{4} \left[\pi - \frac{1}{2} [0 - 0] \right]$

$= \boxed{\frac{\pi}{4}}$