Surface Integrals

Evaluating a Surface Integral

Let S be a surface with equation z = g(x, y) and let R be its projection unto the xy-plane. If g, g_x, and g_y are continuous on R and f is continuous on S, then the surface integral of f over S is:

$$\iint_S f(x, y, z)\,dS$$
$$= \iint_R f(x, y, g(x, y))\sqrt{1+[g_x(x, y)]^2+[g_y(x, y)]^2}\,dA$$

1. Evaluate the surface integral
(Similar to p.1122 #1-4)

$$\iint_S (x-5y+z)\,dS$$
$$S: z = 10 - x,\quad 0 \le x \le 10,\quad 0 \le y \le 10$$

2. Evaluate the surface integral
(Similar to p.1122 #1-4)

$$\iint_S (x-4y+z)\,dS$$
$$S: z = 4, x^2 + y^2 \le 36$$

3. Evaluate the surface integral
(Similar to p.1122 #5-6)

$$\iint_S (4xy)\,dS$$
$$S: z = 5 - x - y, \text{first octant}$$

4. Find the mass of the surface lamina S of density ρ
(Similar to p.1122 #11-12)

$$S: z = \sqrt{9 - x^2 - y^2}, \rho(x, y, z) = kz$$
$$Formula: m = \iint_S \rho(x, y, z)\,dS$$

Surface Integrals (parametric form)

$$dS = \left\| r_u(u,v) \times r_v(u,v) \right\|$$

$$\iint_S f(x(u,v), y(u,v), z(u,v))\, dS$$

5. Evaluate the integral
(Similar to p.1122 #13-16)

$$\iint_S f(x,y)dS$$

where

$$f(x,y) = x + y$$

$$S: r(u,v) = (8\cos u)i + (8\sin u)j + (v)k$$

$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 4$$

6. Evaluate the integral
(Similar to p.1122 #17-22)

$$\iint_S f(x,y,z)dS$$

where

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$S: z = x + y, \quad x^2 + y^2 \le 4$$

Evaluating a Flux Integral

Let S be an oriented surface given by z = g(x, y) and let R be its projection onto the xy-plane

Oriented upward

$$\iint_S F \cdot N\, dS = \iint F \cdot \left[-g_x(x,y)i - g_y(x,y)j + k \right] dA$$

Oriented downward

$$\iint_S F \cdot N\, dS = \iint F \cdot \left[g_x(x,y)i + g_y(x,y)j - k \right] dA$$

7. Find the flux of F through S where N is the upward unit normal vector to S
(Similar to p.1122 #23-28)

$$\iint_S F \cdot N\, dS$$

$$F(x,y,z) = 8zi - 8j + yk$$

$$S: x + y + z = 5, \text{ first octant}$$

Summary: Surface Integrals [z = g(x,y)]

$$dS = \sqrt{1 + \left[g_x(x,y) \right]^2 + \left[g_y(x,y) \right]^2}$$

$$\iint_S f(x,y,z)dS = \iint_R f(x,y,g(x,y))\sqrt{1 + \left[g_x(x,y) \right]^2 + \left[g_y(x,y) \right]^2}\, dA$$

$$\iint_S F \cdot N\, dS = \iint_R F \cdot \left[-g_x(x,y)i - g_y(x,y)j + k \right] dA$$

Summary: Surface Integrals [parametric form]

$$dS = \left\| r_u(u,v) \times r_v(u,v) \right\| dA$$

$$\iint_S f(x,y,z)\,dS = \iint_D f\big(x(u,v),\,y(u,v),\,z(u,v)\big)\,dS$$

$$\iint_S F \cdot N\; dS = \iint_D F \cdot \left(r_u \times r_v \right) dA$$