Trigonometry
Chapter 3 Test Review Key

1. Find the exact value of $\csc\left(\tan^{-1} 7\right)$

Step 1: Set what is inside the parenthesis equal to P

$$P = \tan^{-1} 7$$

Step 2: Eliminate the inverse trig function

$$\tan P = \tan(\tan^{-1} 7)$$

$$\tan P = 7$$

Step 3: Determine the quadrant

inverse tangent is valid in QI and QIV and tangent is positive in QI and QIII so Q1

Step 4: Using x, y, and r formulas; find the missing variable

$$\tan P = \frac{7}{1} \quad and \quad \tan P = \frac{y}{x}$$

$$so \quad y = 7, x = 1$$

Both are positive since we are in quadrant I, now find r:

$$r^2 = x^2 + y^2$$

$$r^2 = 1^2 + 7^2$$

$$r^2 = 1 + 49$$

$$r^2 = 50$$

$$r = \sqrt{50}$$

$$r = \sqrt{5 \cdot 5 \cdot 2}$$

$$r = 5\sqrt{2}$$

Note: we didn't put the plus/minus since r has to be positive

Step 5: Now use x, y, and r formulas to find the outer trig function of original problem

$$\csc = \frac{r}{y}$$

$$\csc = \frac{5\sqrt{2}}{7}$$

$$so$$

$$\csc\left(\tan^{-1} 7\right) = \frac{5\sqrt{2}}{7}$$

2. Verify the identity: $5\cos^2\theta + 3\sin^2\theta = 3 + 2\cos^2\theta$

$$5\cos^2\theta + 3\sin^2\theta = 3 + 2\cos^2\theta$$
$$lets\quad work\quad with\quad left\quad side:$$
$$5\cos^2\theta + 3\sin^2\theta$$
$$= 5\cos^2\theta + 3(1-\cos^2\theta)$$
$$= 5\cos^2\theta + 3 - 3\cos^2\theta$$
$$= 3 + 2\cos^2\theta$$

which is equal to our right side, so we verified the identity

3. Verify the identity: $\frac{\csc\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin^3\theta}$

$$\frac{\csc\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin^3\theta}$$
$$lets\quad work\quad with\quad left\quad side$$
$$\frac{\csc\theta}{1-\cos\theta}$$
$$= \frac{\csc\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}$$
$$= \frac{\csc\theta(1+\cos\theta)}{1-\cos^2\theta}$$
$$= \frac{\csc\theta(1+\cos\theta)}{\sin^2\theta}$$
$$= \frac{\frac{1}{\sin\theta}(1+\cos\theta)}{\sin^2\theta}$$
$$= \frac{\sin\theta\cdot\frac{1}{\sin\theta}(1+\cos\theta)}{\sin\theta\cdot\sin^2\theta}$$
$$= \frac{1+\cos\theta}{\sin^3\theta}$$

which equals our right side so we verified the identity

4. Verify the identity: $\csc\theta - \sin\theta = \cos\theta\cot\theta$

$$\csc\theta - \sin\theta = \cos\theta\cot\theta$$

lets work with left side

$$\csc\theta - \sin\theta$$

$$= \frac{1}{\sin\theta} - \sin\theta$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta}{1}$$

$$= \frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}$$

$$= \frac{1 - \sin^2\theta}{\sin\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta}$$

$$= \frac{\cos\theta}{1} \cdot \frac{\cos\theta}{\sin\theta}$$

$$= \cos\theta\cot\theta$$

which is equal to our right side so we verified the identity

5. Find the exact value of $\sin\left(\frac{-\pi}{12}\right)$ using a) sum and difference formulas and b) half-angle formulas

a)	b)
$\sin\left(\frac{-\pi}{12}\right)$ $=\sin\left(\frac{2\pi}{12}-\frac{3\pi}{12}\right)$ $=\sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)$ $so \quad \alpha=\frac{\pi}{6}, \beta=\frac{\pi}{4}$ $and \quad \sin(\alpha-\beta)$ $=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $=\sin\frac{\pi}{6}\cos\frac{\pi}{4}-\cos\frac{\pi}{6}\sin\frac{\pi}{4}$ $=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ $=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$ $=\frac{\sqrt{2}-\sqrt{6}}{4}$	$\sin\left(\frac{-\pi}{12}\right)=-\sin\left(\frac{\pi}{12}\right)$ and $\frac{\pi}{12}$ is in QI so half the angle is also in QI $-\sin\left(\frac{\pi}{12}\right)$ $=-\sin\left(\frac{\pi/6}{2}\right)$ $so \quad \alpha=\frac{\pi}{6}$ $and \quad \sin\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{2}}(in \quad QI)$ $=-\sqrt{\frac{1-\cos\frac{\pi}{6}}{2}}$ $=-\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$ $=-\sqrt{\frac{2(1)-2\left(\frac{\sqrt{3}}{2}\right)}{2(2)}}$ $=-\sqrt{\frac{2-\sqrt{3}}{4}}$ $=-\frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}}$ $=-\frac{\sqrt{2-\sqrt{3}}}{2}$

6. Given $\tan\theta = \frac{8}{3}, \pi < \theta < \frac{3\pi}{2}$, find a) $\sin(2\theta)$ and b) $\sin\frac{\theta}{2}$

$\tan\theta = \frac{8}{3}$ and it is in Quadrant III so y = -8 and x = -3, so now let's find r:

$$r^2 = x^2 + y^2$$
$$r^2 = (-3)^2 + (-8)^2$$
$$r^2 = 9 + 64$$
$$r^2 = 73$$
$$r = \sqrt{73}$$

Now let's find sin and cos:

$$\sin\theta = \frac{y}{r} = \frac{-8}{\sqrt{73}} = \frac{-8\sqrt{73}}{73}$$
$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{73}} = \frac{-3\sqrt{73}}{73}$$

a)

$$\sin(2\theta)$$
$$= 2\sin\theta\cos\theta$$
$$= 2\left(\frac{-8\sqrt{73}}{73}\right)\left(\frac{-3\sqrt{73}}{73}\right)$$
$$= \frac{48\cdot 73}{5329}$$
$$= \frac{3504}{5329}$$
$$= \frac{48}{73}$$

b)

theta is in QIII so theta/2 is in QII and sin is positive there so:

$$\sin\frac{\theta}{2}$$
$$= \sqrt{\frac{1-\cos\theta}{2}}$$
$$= \sqrt{\frac{1-\frac{-3\sqrt{73}}{73}}{2}}$$
$$= \sqrt{\frac{73(1)+73\left(\frac{3\sqrt{73}}{73}\right)}{2(73)}}$$
$$= \sqrt{\frac{73+3\sqrt{73}}{146}}$$
$$= \frac{\sqrt{73+3\sqrt{73}}}{\sqrt{146}}$$
$$= \frac{\sqrt{146(73+3\sqrt{73})}}{146}$$

7. Express each product as a sum containing only sines or cosines: $\cos(3\theta)\cos(4\theta)$

$\cos(3\theta)\cos(4\theta)$

$so \quad \alpha = 3\theta, \beta = 4\theta$

$and \quad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

$= \frac{1}{2}\left[\cos(3\theta-4\theta)+\cos(3\theta+4\theta)\right]$

$= \frac{1}{2}\left[\cos(-\theta)+\cos(7\theta)\right]$

$= \frac{1}{2}\left[\cos(\theta)+\cos(7\theta)\right]$

Problems 8-10 are from 0 to 2π and find the exact answer(s)

8. Solve: $2\cos\theta+\sqrt{2}=0$

$2\cos\theta+\sqrt{2}=0$

$2\cos\theta=-\sqrt{2}$

$\cos\theta=\frac{-\sqrt{2}}{2}$

$so \quad \theta=\frac{3\pi}{4},\frac{5\pi}{4}$

9. Solve: $2\sin^2\theta-3\sin\theta+1=0$

$2\sin^2\theta-3\sin\theta+1=0$

$(2\sin\theta-1)(\sin\theta-1)=0$

$2\sin\theta-1=0$	$\sin\theta-1=0$
$2\sin\theta=1$	$\sin\theta=1$
$\sin\theta=\frac{1}{2}$	$\theta=\frac{\pi}{2}$
$\theta=\frac{\pi}{6},\frac{5\pi}{6}$	

10. Solve: $\cos\theta = \sec\theta$

$$\cos\theta = \sec\theta$$

$$\cos\theta = \frac{1}{\cos\theta}$$

$$\cos\theta(\cos\theta) = \cos\theta\left(\frac{1}{\cos\theta}\right)$$

$$\cos^2\theta = 1$$

$$\cos\theta = \pm\sqrt{1}$$

$$\cos\theta = \pm 1$$

$$\theta = 0, \pi$$